OBJECTS AS CONSTRAINTS

A Formalism of Order-Sorted Featured Structures

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OUTLINE

- Motivation and background
- Basic order-sorted feature ($\mathcal{OSF}$) formalism
- Disjunction and negation
- Partial features, extensional sorts, relational features (aggregation)
- $\mathcal{OSF}$ theory unification
- Conclusion: A few fundamental principles...

Motivation

- **Proposal:** a formalism for representing objects that is: intuitive (objects as labelled graphs), expressive ("real-life" data models), formal (logical semantics), operational (executable), and efficient (constraint-solving)
- **Why?** *viz.*, ubiquitous use of labelled graphs to structure information naturally as in:
  - object-orientation, knowledge representation,
  - databases, constraint-based programming,
  - natural language processing, graphical interfaces,
  - concurrency and communication,
  - XML, RDF, "Semantic Web," *etc.*, ...

BACKGROUND

This work is the synthesis of research of many years by many people:

- Hassan Aït-Kaci (since 1983)
- Gert Smolka (since 1986)
- Andreas Podelski (since 1989)
- Franz Baader, Rolf Backhofen, Jochen Dörre, Martin Emele, Bernhard Nebel, Joachim Niehren, Ralf Treinen, Manfred Schmidt-Schauß, Remi Zajac, . . .

$\mathcal{OSF}$ **signature:**

$$\langle \mathcal{S}, \leq, \wedge, \mathcal{F} \rangle$$

s.t.:

- $\mathcal{S}$ is a set of sorts containing the sorts $\top$ and $\bot$
- $\leq$ is a partial order on $\mathcal{S}$ ($\bot$ is least element, $\top$ is greatest element)
- $\langle \mathcal{S}, \leq, \wedge \rangle$ is a lower semi-lattice ($s \wedge s'$ is called the greatest common subsort of s and s')
- $\mathcal{F}$ is a set of feature symbols.

Given an $\mathcal{OSF}$ signature $\langle \mathcal{S}, \leq, \wedge, \mathcal{F} \rangle$ an $\mathcal{OSF}$ algebra is a structure:

$$\mathfrak{A} = \langle D^{\mathfrak{A}}, (s^{\mathfrak{A}})_{s \in \mathcal{S}}, (\ell^{\mathfrak{A}})_{\ell \in \mathcal{F}} \rangle$$

s.t.:

- $D^{\mathfrak{A}} \neq \emptyset$ is a set: the domain of $\mathfrak{A}$
- $s^{\mathfrak{A}} \subseteq D^{\mathfrak{A}}$ for s in $\mathcal{S}$ $(\top^{\mathfrak{A}} = D^{\mathfrak{A}}, \ \perp^{\mathfrak{A}} = \emptyset)$
- $(s \wedge s')^{\mathfrak{A}} = s^{\mathfrak{A}} \cap s'^{\mathfrak{A}}$
- $\ell^{\mathfrak{A}} : D^{\mathfrak{A}} \mapsto D^{\mathfrak{A}}$ for ℓ in $\mathcal{F}$ (*i.e.*, $\ell^{\mathfrak{A}}$ is a (total) function from the domain to the domain)

$\mathcal{OSF}$ HOMOMORPHISM

$\mathcal{OSF}$ homomorphism between two $\mathcal{OSF}$ algebras $\mathfrak{A}$ and $\mathfrak{B}$:

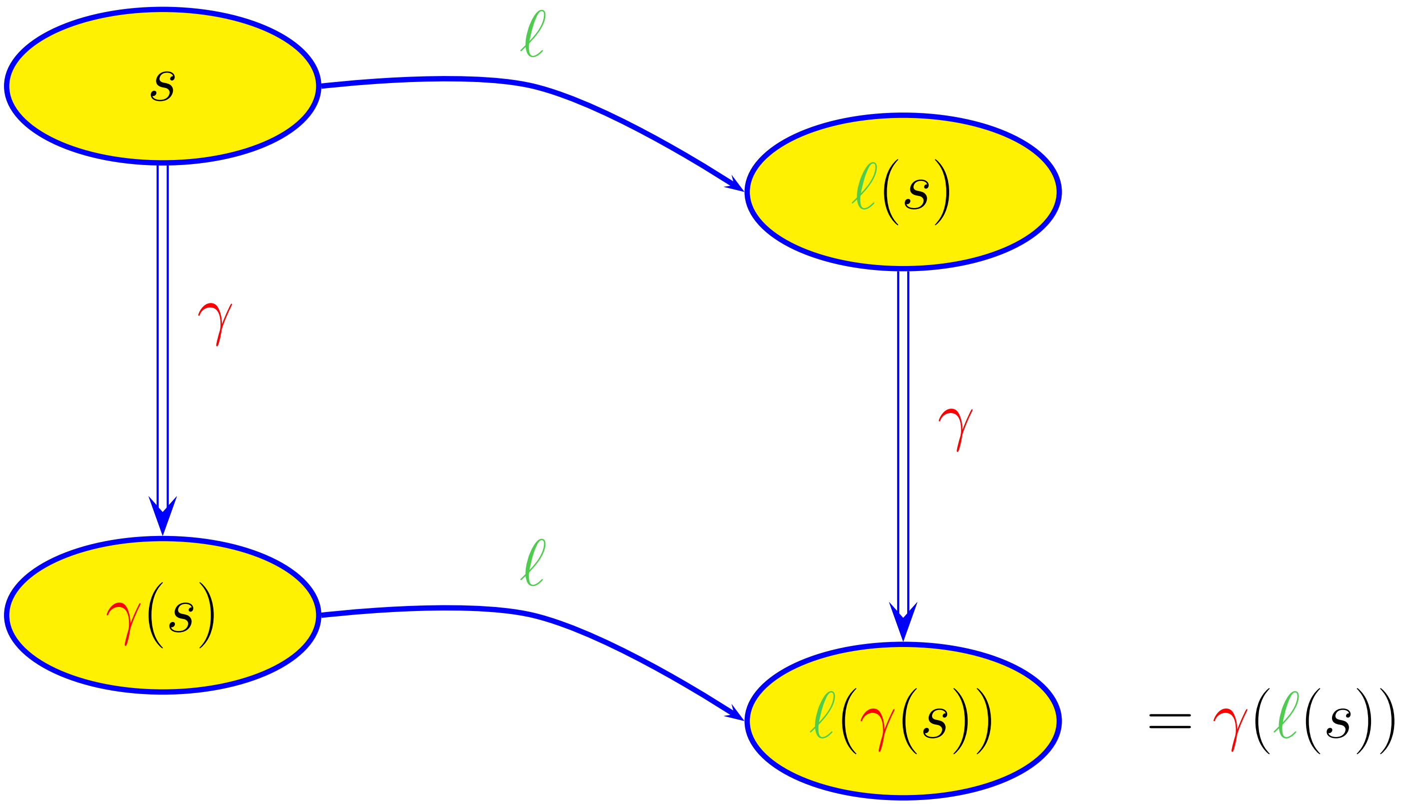
- $\gamma : D^{\mathfrak{A}} \mapsto D^{\mathfrak{B}}$
- $\gamma(\ell^{\mathfrak{A}}(d)) = \ell^{\mathfrak{B}}(\gamma(d))$ for all $d \in D^{\mathfrak{A}}$
- $\gamma(s^{\mathfrak{A}}) \subseteq s^{\mathfrak{B}}$ for all $s \in \mathcal{S}$

Taking $\mathfrak{A} = \mathfrak{B}$, γ is an $\mathcal{OSF}$ endomorphism of $\mathfrak{A}$:

- $\forall d \in D, \;\; \gamma(\ell(d)) = \ell(\gamma(d))$
- $\forall s \in \mathcal{S}, \;\; \gamma(s) \subseteq s$

This definition captures exactly inheritance of attributes.

INHERITANCE = $\mathcal{OSF}$ ENDOMORPHISM



Hence, inheritance is endomorphic approximation.

$\mathcal{OSF}$ TERM SYNTAX

Let $\mathcal{V}$ be a countably infinite set of variables.

An $\mathcal{OSF}$ term is an expression of the form:

$$X : s(\ell_1 \Rightarrow t_1, \ldots, \ell_n \Rightarrow t_n)$$

where:

- $X \in \mathcal{V}$ is the root variable
- $s \in \mathcal{S}$ is the root sort
- $n \geq 0$ (if $n = 0$, we write $X : s$)
- $\{\ell_1, \ldots, \ell_n\} \subseteq \mathcal{F}$ are features
- $t_1, \ldots, t_n$ are $\mathcal{OSF}$ terms

Example

$$
\begin{aligned}
X : person(&name \Rightarrow N : \top(first \Rightarrow F : string),\\
&name \Rightarrow M : id(last \Rightarrow S : string),\\
&spouse \Rightarrow P : person(name \Rightarrow I : id(last \Rightarrow S : \top),\\
&\qquad\qquad\qquad spouse \Rightarrow X : \top)).
\end{aligned}
$$

Lighter notation:

$$
\begin{aligned}
X : person(&name \Rightarrow \top(first \Rightarrow string),\\
&name \Rightarrow id(last \Rightarrow S : string),\\
&spouse \Rightarrow person(name \Rightarrow id(last \Rightarrow S),\\
&\qquad\qquad\qquad spouse \Rightarrow X)).
\end{aligned}
$$

$\mathcal{OSF}$ TERM SEMANTICS

- $\mathcal{OSF}$ term $t = X : s(\ell_1 \Rightarrow t_1, \ldots, \ell_n \Rightarrow t_n)$
- $\mathcal{OSF}$ interpretation $\mathfrak{A}$
- $\mathfrak{A}$-valuation $\alpha : \mathcal{V} \mapsto D^{\mathfrak{A}}$

Denotation of t in $\mathfrak{A}$ under valuation α:

$$[\![t]\!]^{\mathfrak{A},\alpha} \stackrel{\text{DEF}}{=\!=} \{\alpha(X)\} \cap s^{\mathfrak{A}} \cap \left(\bigcap_{1 \leq i \leq n} (\ell_i^{\mathfrak{A}})^{-1}([\![t_i]\!]^{\mathfrak{A},\alpha}) \right)$$

Denotation of t in $\mathfrak{A}$ under all possible valuations:

$$[\![t]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=\!=} \bigcup_{\alpha:\mathcal{V}\mapsto D^{\mathfrak{A}}} [\![t]\!]^{\mathfrak{A},\alpha}.$$

$\mathcal{OSF}$ CLAUSE SYNTAX

For X and X' variables in $\mathcal{V}$, s a sort in $\mathcal{S}$, and ℓ a feature in $\mathcal{F}$, an $\mathcal{OSF}$ constraint is one of:

- $X : s$
- $X.\ell \doteq X'$
- $X \doteq X'$

An $\mathcal{OSF}$ clause is a conjunction of $\mathcal{OSF}$constraints—*i.e.*, a set of $\mathcal{OSF}$ constraints

- $\phi_1 \quad \& \quad \ldots \quad \& \quad \phi_n$

SEMANTICS OF $\mathcal{OSF}$ CLAUSES

Satisfaction of $\mathcal{OSF}$ constraints in an $\mathcal{OSF}$ algebra $\mathfrak{A}$ by a valuation $\alpha : \mathcal{V} \mapsto D^{\mathfrak{A}}$ is defined by:

- $\mathfrak{A}, \alpha \models X : s$ iff $\alpha(X) \in s^{\mathfrak{A}}$
- $\mathfrak{A}, \alpha \models X \doteq Y$ iff $\alpha(X) = \alpha(Y)$
- $\mathfrak{A}, \alpha \models X.\ell \doteq Y$ iff $\ell^{\mathfrak{A}}(\alpha(X)) = \alpha(Y)$
- $\mathfrak{A}, \alpha \models \phi_1 \;\;\&\;\; \ldots \;\;\&\;\; \phi_n$ iff $\mathfrak{A}, \alpha \models \phi_i \;\; \forall i = 1, \ldots, n$

FROM $\mathcal{OSF}$ TERMS TO $\mathcal{OSF}$ CLAUSES

An $\mathcal{OSF}$ term $t = X : s(\ell_1 \Rightarrow t_1, \ldots, \ell_n \Rightarrow t_n)$

is dissolved into an $\mathcal{OSF}$ clause $\phi(t)$ as follows:

$$\begin{array}{lllllllll} \varphi(t) & \stackrel{\text{DEF}}{=} & X : s & \& & X.f_1 \doteq X_1 & \& & \ldots & \& & X.f_n \doteq X_n \\ & & & \& & \varphi(t_1) & \& & \ldots & \& & \varphi(t_n) \end{array}$$

where $X_1, \ldots, X_n$ are the root variables of $t_1, \ldots, t_n$.

Theorem: $\mathfrak{A}, \alpha \models \varphi(t) \iff [\![t]\!]^{\mathfrak{A},\alpha} \neq \emptyset$

Example of $\mathcal{OSF}$ term dissolution

$$
\begin{aligned}
t \;=\; X : person(&name \Rightarrow N : \top(first \Rightarrow F : string),\\
&name \Rightarrow M : id(last \Rightarrow S : string),\\
&spouse \Rightarrow P : person(name \Rightarrow I : id(last \Rightarrow S : \top),\\
&\qquad\qquad\qquad\qquad spouse \Rightarrow X : \top))
\end{aligned}
$$

$$
\begin{aligned}
\varphi(t) = X : person \quad & \& \quad X.name \doteq N \quad \& \quad N : \top \\
& \& \quad X.name \doteq M \quad \& \quad M : id \\
& \& \quad X.spouse \doteq P \quad \& \quad P : person \\
& \& \quad N.first \doteq F \quad \& \quad F : string \\
& \& \quad M.last \doteq S \quad \& \quad S : string \\
& \& \quad P.name \doteq I \quad \& \quad I : id \\
& \& \quad I.last \doteq S \quad \& \quad S : \top \\
& \& \quad P.spouse \doteq X \quad \& \quad X : \top
\end{aligned}
$$

Basic $\mathcal{OSF}$ term normalization

(1) Sort Intersection

$$\frac{\phi \ \& \ X : s \ \& \ X : s'}{\phi \ \& \ X : s \wedge s'}$$

(3) Variable Elimination

$$\frac{\phi \ \& \ X \doteq X'}{\phi[X'/X] \ \& \ X \doteq X'}$$

if $X \neq X'$
and $X \in \textbf{\textit{Var}}(\phi)$

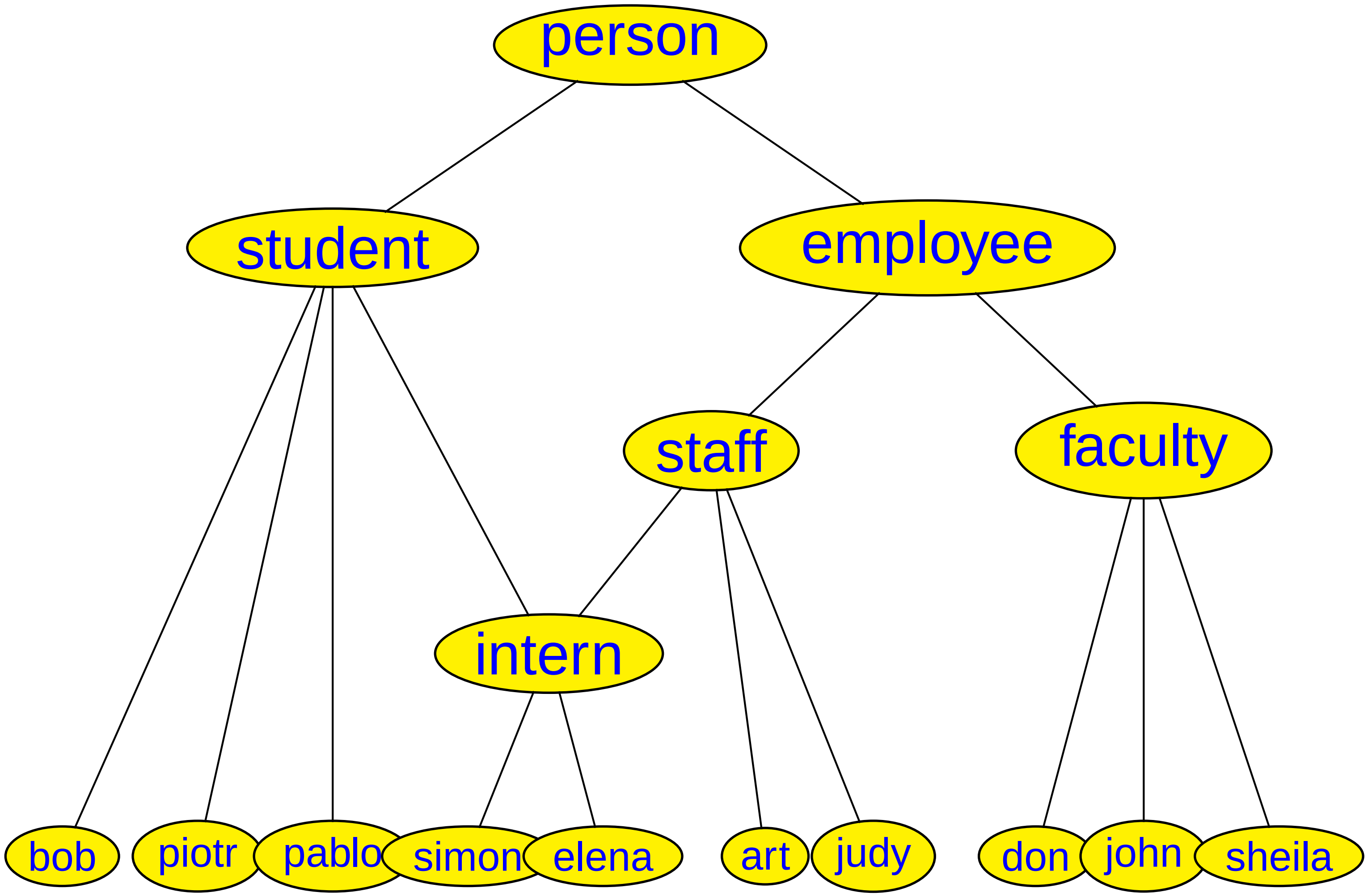
(2) Inconsistent Sort

$$\frac{\phi \ \& \ X : \bot}{X : \bot}$$

(4) Feature Functionality

$$\frac{\phi \ \& \ X.\ell \doteq X' \ \& \ X.\ell \doteq X''}{\phi \ \& \ X.\ell \doteq X' \ \& \ X' \doteq X''}$$

$\mathcal{OSF}$ TERM UNIFICATION = $\mathcal{OSF}$ TERM NORMALIZATION



$\mathcal{OSF}$ TERM UNIFICATION = $\mathcal{OSF}$ TERM NORMALIZATION

```
X : student
    (roommate => person(rep => E : employee),
     advisor  => don(secretary => E))

&

Y : employee
    (advisor  => don(assistant => A),
     roommate => S : student(rep => S),
     helper   => simon(spouse => A))

&

X = Y
```

$\mathcal{OSF}$ TERM UNIFICATION = $\mathcal{OSF}$ TERM NORMALIZATION

```
X : intern
    (advisor => don(assistant => A,
                    secretary => S),
     helper => simon(spouse => A),
     roommate => S : intern(rep => S))

&

X = Y

&

E = S
```

EXTENDED $\mathcal{OSF}$ TERMS

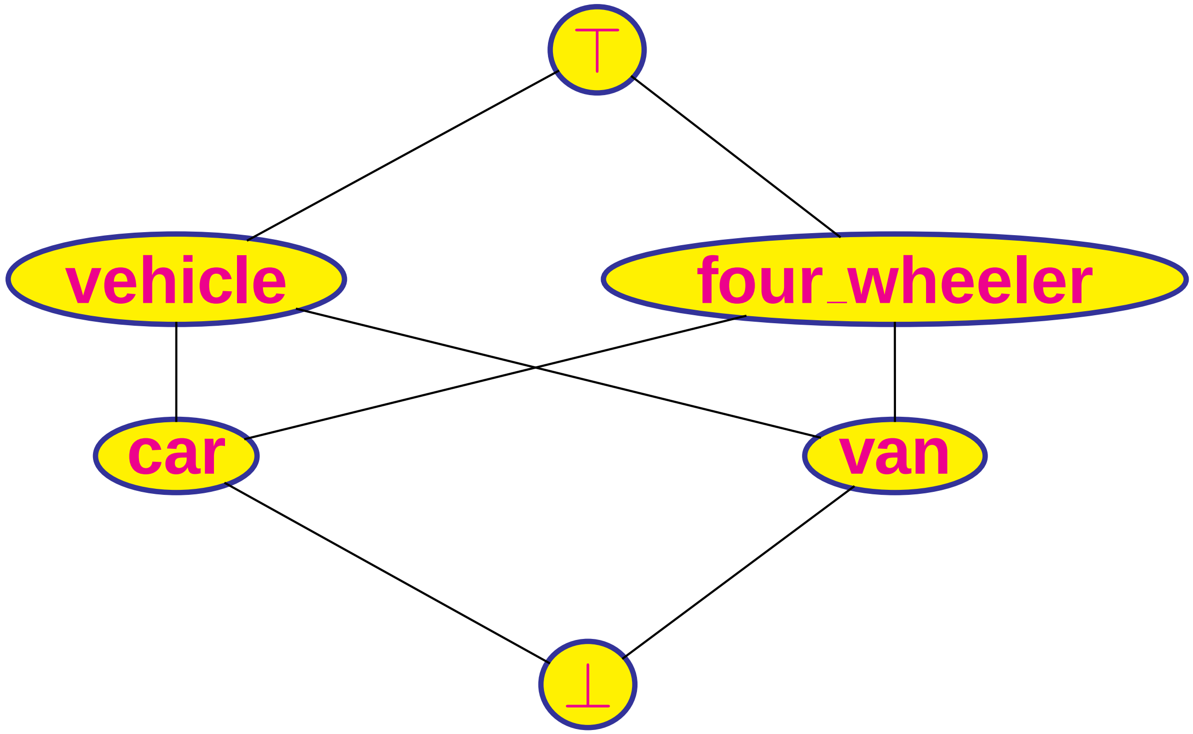
Basic $\mathcal{OSF}$ terms may be extended to express:

- Non-lattice sort signatures
- Disjunction
- Negation
- Partial features
- Extensional sorts (*i.e.*, denoting elements)
- Relational features (*a.k.a.*, "roles")
- Aggregates
- Sort definitions (*a.k.a.*, "$\mathcal{OSF}$ theories")

EXTENDED $\mathcal{OSF}$ TERMS

$$
\begin{aligned}
\texttt{OsfTerm} &::= [\,\texttt{Variable:}\,]\ \texttt{Term} \\
\texttt{Term} &::= \texttt{ConjunctiveTerm} \\
&\mid \texttt{DisjunctiveTerm} \\
&\mid \texttt{NegativeTerm} \\
\texttt{ConjunctiveTerm} &::= \texttt{Sort}\,[\,(\,\texttt{Attribute}^{+}\,)\,] \\
\texttt{Attribute} &::= \texttt{Feature} \Rightarrow \texttt{OsfTerm} \\
\texttt{DisjunctiveTerm} &::= \{\,\texttt{OsfTerm}\ [\,;\ \texttt{OsfTerm}\,]^{*}\,\} \\
\texttt{NegativeTerm} &::= \neg\,\texttt{OsfTerm}
\end{aligned}
$$

ENABLING NON-LATTICE SIGNATURES USING DISJUNCTION



Non-unique GLBs are disjunctive sorts:

$$\textbf{vehicle} \wedge \textbf{four\_wheeler} = \{\textbf{car}; \textbf{van}\}$$

DISJUNCTIVE $\mathcal{OSF}$ TERMS

Syntax of disjunctive $\mathcal{OSF}$ terms:

$$\{\ t_1\ ;\ \ldots\ ;\ t_n\ \}$$

Semantics of disjunctive $\mathcal{OSF}$ terms:

$$[\![\{t_1; \ldots ; t_n\}]\!]^{\mathfrak{A},\alpha} \stackrel{\text{DEF}}{=\!=} \bigcup_{1 \leq i \leq n} [\![t_i]\!]^{\mathfrak{A},\alpha}$$

Disjunctive $\mathcal{OSF}$ clauses:

$$\varphi(\{t_1; \ldots ; t_n\}) \stackrel{\text{DEF}}{=\!=} \varphi(t_1) \parallel \ldots \parallel \varphi(t_n)$$

$\mathfrak{A}, \alpha \models \phi_1 \parallel \ldots \parallel \phi_n$ iff $\mathfrak{A}, \alpha \models \phi_i$ for some $i = 1, \ldots, n$

DISJUNCTIVE $\mathcal{OSF}$ NORMALIZATION

(5) Non-unique GLB

$$\frac{\phi \quad \& \quad X : s \quad \& \quad X : s'}{\phi \quad \& \quad (X : s_1 \parallel \ldots \parallel X : s_n)}$$

where $\{s_i\}_{i=0}^{n} \stackrel{\text{DEF}}{=} \max_{\leq}\{t \in \mathcal{S} \mid t \leq s \text{ and } t \leq s'\}$

(6) Distributivity

$$\frac{\phi \quad \& \quad (\phi' \parallel \phi'')}{(\phi \quad \& \quad \phi') \quad \parallel \quad (\phi \quad \& \quad \phi'')}$$

(7) Disjunction

$$\frac{\phi \parallel \phi'}{\phi}$$

NEGATION

Syntax of negative $\mathcal{OSF}$ terms: $\neg t$

Semantics of negative $\mathcal{OSF}$ terms: $[\![\neg t]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} D^{\mathfrak{A}} \setminus [\![t]\!]^{\mathfrak{A}}$

Complemented sorts: $[\![\overline{s}]\!]^{\mathfrak{A}} \stackrel{\text{DEF}}{=} D^{\mathfrak{A}} \setminus [\![s]\!]^{\mathfrak{A}}$

Sorted variable simplification:

$$\varsigma(X : s) \stackrel{\text{DEF}}{=} X : s \qquad \text{if } s \in \mathcal{S}$$

$$\varsigma(X : \overline{\overline{s}}) \stackrel{\text{DEF}}{=} \varsigma(X : s)$$

$$\varsigma(X : \overline{\{s_1; \ldots; s_n\}}) \stackrel{\text{DEF}}{=} \varsigma(X : s_1) \ \& \ \ldots \ \& \ \varsigma(X : s_n)$$

NEGATIVE $\mathcal{OSF}$ TERMS

Dissolving negative $\mathcal{OSF}$ terms into $\mathcal{OSF}$ clauses eliminates negation:

$$\varphi(\neg(\neg t)) \stackrel{\text{DEF}}{=\!=} \varphi(t)$$

$$\varphi(\neg\{t_1; \ldots ; t_n\}) \stackrel{\text{DEF}}{=\!=} \varphi(\neg t_1) \ \& \ \ldots \ \& \ \varphi(\neg t_n)$$

$$\begin{array}{rcl} \varphi(\neg X : s(\ell_i \Rightarrow t_i)_{i=1}^{n}) & \stackrel{\text{DEF}}{=\!=} & \varsigma(X : \overline{s}) \\ & \| & X.\ell_1 \doteq X_1 \ \& \ \varphi(\neg t_1) \\ & \| & X.\ell_1 \doteq X_1' \ \& \ X_1' \dot{\neq} X_1 \ \& \ \varphi(t_1) \\ & & \ldots \\ & \| & X.\ell_n \doteq X_n \ \& \ \varphi(\neg t_n) \\ & \| & X.\ell_n \doteq X_n' \ \& \ X_n' \dot{\neq} X_n \ \& \ \varphi(t_n) \end{array}$$

NEGATIVE $\mathcal{OSF}$ TERM NORMALIZATION

(8) Variable Disequality

$$\frac{\phi \;\&\; X \dot{\neq} X}{\bot}$$

(9) Sort Complement

$$\frac{\phi \;\&\; X : \overline{s}}{\phi \;\&\; X : s'}$$

if $s' \in \max_{\leq}\{t \in \mathcal{S} \mid s \not\leq t$ and $t \not\leq s\}$

PARTIAL FEATURES

Partial features have restricted domains:

$$\exists y, \quad y = \ell(x) \text{ only if } x \in \textbf{\textit{Dom}}(\ell)$$

Declaring partial feature domains:

$$\textbf{\textit{Dom}} : \mathcal{F} \mapsto \mathbf{2}^{\mathcal{S}}$$

s.t. $\textbf{\textit{Dom}}(\ell) \stackrel{\text{DEF}}{=\!=}$ set of maximal sorts where ℓ is defined. Can also declare a feature's range: $\textbf{\textit{Ran}}_s : \mathcal{F} \mapsto \mathcal{S}$ for $s \in \textbf{\textit{Dom}}(\ell)$.

(10) Partial Feature

$$\frac{\phi \;\&\; X.\ell \doteq X'}{\phi \;\&\; X.\ell \doteq X' \;\&\; X : s \;\&\; X' : s'} \quad \begin{array}{ll} \text{if} & s \in \textbf{\textit{Dom}}(\ell) \\ \text{and} & \textbf{\textit{Ran}}_s(\ell) = s' \end{array}$$

PARTIAL FEATURES (EXAMPLE)

Assume $\{nil, cons, list\} \subseteq \mathcal{S}$ such that:

$$\begin{array}{lcl} nil & < & list \\ cons & < & list \end{array}$$

and $\{hd, tl\} \subseteq \mathcal{F}$ such that:

$$\begin{array}{lcl} \textbf{\textit{Dom}}(hd) & \stackrel{\text{DEF}}{=\!=} & \{cons\} \\ \textbf{\textit{Dom}}(tl) & \stackrel{\text{DEF}}{=\!=} & \{cons\} \end{array}$$

then:

$$\begin{array}{lcl} list(tl \Rightarrow X) & \leadsto & cons(tl \Rightarrow X) \\ int(tl \Rightarrow X) & \leadsto & \bot \end{array}$$

EXTENSIONAL SORTS

The fact that some sorts denote singletons (*e.g.*, numbers) is not part of our axioms so far!

i.e.,

$$f(a \Rightarrow 1, b \Rightarrow 1) \not\leq f(a \Rightarrow X, b \Rightarrow X)$$

because:

$$f(a \Rightarrow X : s, b \Rightarrow X' : s) \leq f(a \Rightarrow Y, b \Rightarrow Y) \quad \text{iff} \quad X = X'$$

A sort that denotes a singleton, whenever all its images by a specific set of features do, is called extensional.

EXTENSIONAL SORTS

Extensional sorts are element constructors.

Let $\mathcal{E} \subseteq Minimals(\mathcal{S})$ be the set of extensional sorts with rank function:

$$\textbf{\textit{Arity}} : \mathcal{E} \mapsto \mathbf{2}^{\mathcal{F}}$$

$$\begin{aligned} \textit{e.g.} : \quad \textbf{\textit{Arity}}(n) &= \emptyset \quad \forall n \in \mathbb{N} \\ \textbf{\textit{Arity}}(nil) &= \emptyset \\ \textbf{\textit{Arity}}(cons) &= \{hd, tl\} \end{aligned}$$

EXTENSIONAL SORTS

Extensional sorts obey an axiom reminiscent of the axiom of functionality; *viz.*,

$$\text{if} \quad \textbf{\textit{Arity}}(f) = n \ \text{ and } \ X_i = Y_i \ \ (\forall i = 1, \ldots, n)$$

$$\text{then} \ f(X_1, \ldots, X_n) \ = \ f(Y_1, \ldots, Y_n)$$

(11) Weak Extensionality

$$\frac{\phi \ \& \ X : s \ \& \ X' : s}{\phi \ \& \ X : s \ \& \ X \doteq X'}$$

if $s \in \mathcal{E}$ and $\forall \ell \in \textbf{\textit{Arity}}(s)$:

$$\{X.f \doteq Y, X'.f \doteq Y\} \subseteq \phi$$

EXTENSIONAL SORTS

The Weak Extensionality rule works, but not for cyclic terms; *viz.*:

$$\text{let } s \in \mathcal{E} \text{ and } \textbf{\textit{Arity}}(s) = \{\ell\}$$

$$\text{then } X : s(\ell \Rightarrow X) \;\&\; X' : s(\ell \Rightarrow X')$$

$$\text{or } X : s(\ell \Rightarrow X') \;\&\; X' : s(\ell \Rightarrow X)$$

are not reduced! So we need a stronger condition for cycles.

STRONG EXTENSIONALITY

Proceed coinductively from roots to leaves carrying a context Γ, a set of pairs $s/\{X_1, \ldots, X_n\}$ s.t. $X_i \in \mathcal{V}$ $(i = 1, \ldots, n)$ and $s \in \mathcal{E}$ occurs at most once in Γ:

(12) Extensional Occurrence

$$\frac{\Gamma \uplus \{s/V, \ldots, \} \vdash \phi \;\; \& \;\; X : s}{\Gamma \uplus \{s/V \cup \{X\}, \ldots\} \vdash \phi \;\; \& \;\; X : s}$$

if $s \in \mathcal{E}$ and $X \notin V$ and $\forall f \in \textit{\textbf{Arity}}(s)$: $\{X.f \doteq X', X' : s'\} \subseteq \phi$ with $s' \in \mathcal{E}$

(13) Strong Extensionality

$$\frac{\Gamma \uplus \{s/\{X, X', \ldots\}\} \vdash \phi}{\Gamma \uplus \{s/\{X, \ldots\}\} \vdash \phi \;\; \& \;\; X \doteq X'}$$

if $s \in \mathcal{E}$

FIRST-ORDER TERMS AS $\mathcal{OSF}$ TERMS

Let $\Sigma \stackrel{\text{DEF}}{=} \biguplus_{n\in\mathbb{N}} \Sigma_n$ be a ranked signature.

The first-order (rational) terms in $\mathcal{T}_{\Sigma,\mathcal{V}}$ are $\mathcal{OSF}$ terms s.t.:

- $\mathcal{S} \stackrel{\text{DEF}}{=} \Sigma \cup \{\top, \bot\}$ is a flat lattice
- $\mathcal{F} \stackrel{\text{DEF}}{=} \mathbb{N} \setminus \{0\}$
- $\textbf{\textit{Arity}}(\top) \stackrel{\text{DEF}}{=} \emptyset$
- $\textbf{\textit{Arity}}(\bot) \stackrel{\text{DEF}}{=} \{i \in \mathbb{N}^* \mid i \leq \max\{n > 0 \mid \Sigma_n \neq \emptyset\}\ \}$
- $\forall f \in \Sigma_n : \ \ \textbf{\textit{Arity}}(f) \stackrel{\text{DEF}}{=} \{1, \ldots, n\}$
- $\forall i \in \mathcal{F} : \ \ \textbf{\textit{Dom}}(i) \stackrel{\text{DEF}}{=} \bigcup_{i \leq n} \Sigma_n$
- $\forall i \in \mathcal{F}, \forall f \in \Sigma : \ \ \textbf{\textit{Ran}}_f(i) \stackrel{\text{DEF}}{=} \begin{cases} \top & \text{if } f \in \textbf{\textit{Dom}}(i) \\ \bot & \text{otherwise} \end{cases}$

RELATIONAL FEATURES AND AGGREGATION

Relational features are set-valued features:

$$\forall \langle x, y \rangle \in A \times B : \quad \langle x, y \rangle \in R \quad \textbf{iff} \quad y \in R[x] \quad \textbf{iff} \quad x \in R^{-1}[y]$$

Sets are a particular case of monoidal aggregates:

- the notation "$X : s$" is generalized to carry an optional value $e \in \mathcal{E}$
- "$X = e : s$" means "X has value e of sort s" $(X \in \mathcal{V},\ e \in \mathcal{E},\ s \in \mathcal{S})$
- the shorthand "$X = e$" means "$X = e : \top$"
- when the sort $s \in \mathcal{S}$ denotes a commutative monoid $\langle \star, \mathbf{1}_\star \rangle$, the shorthand "$X : s$" means "$X = \mathbf{1}_\star : s$."

The semantic conditions are thus extended:

$$\mathfrak{A}, \alpha \models X = e : s \text{ iff } e^{\mathfrak{A}} \in s^{\mathfrak{A}} \text{ and } \alpha(X) = e^{\mathfrak{A}}$$

(14) Value Aggregation

$$\frac{\phi \;\&\; X = e : s \;\&\; X = e' : s'}{\phi \;\&\; X = e \star e' : s \wedge s'}$$

if s and s' are both subsorts of commutative monoid $\langle \star, \mathbf{1}_\star \rangle$

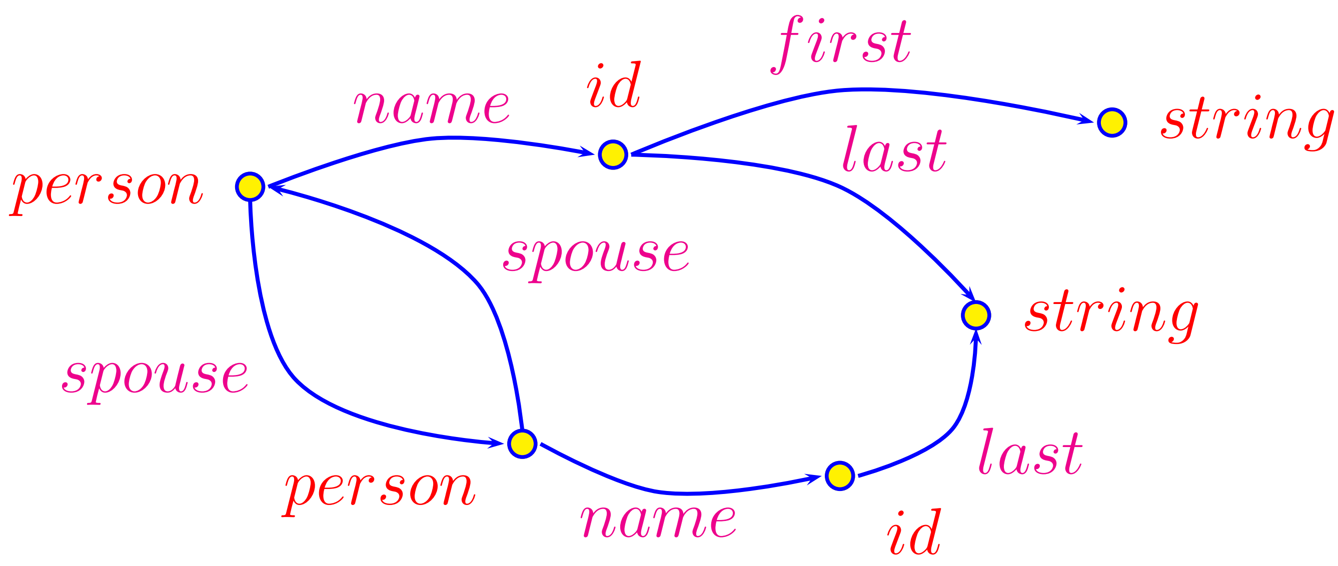
N.B.: This works for any commutative monoid—not just sets!

IDEA: Augment the sort ordering with constraints imposing:

- sorts of features
- coreference equations

e.g., define the sort person to abide by the structure:

$\Theta(person) =$



An $\mathcal{OSF}$ theory is a function: $\Theta : \mathcal{S} \mapsto \Psi$

An $\mathcal{OSF}$ theory is order-consistent iff it is monotonic:

$$s \leq s' \;\Rightarrow\; \Theta(s) \leq \Theta(s')$$

$\mathcal{OSF}$ theory unification problem:

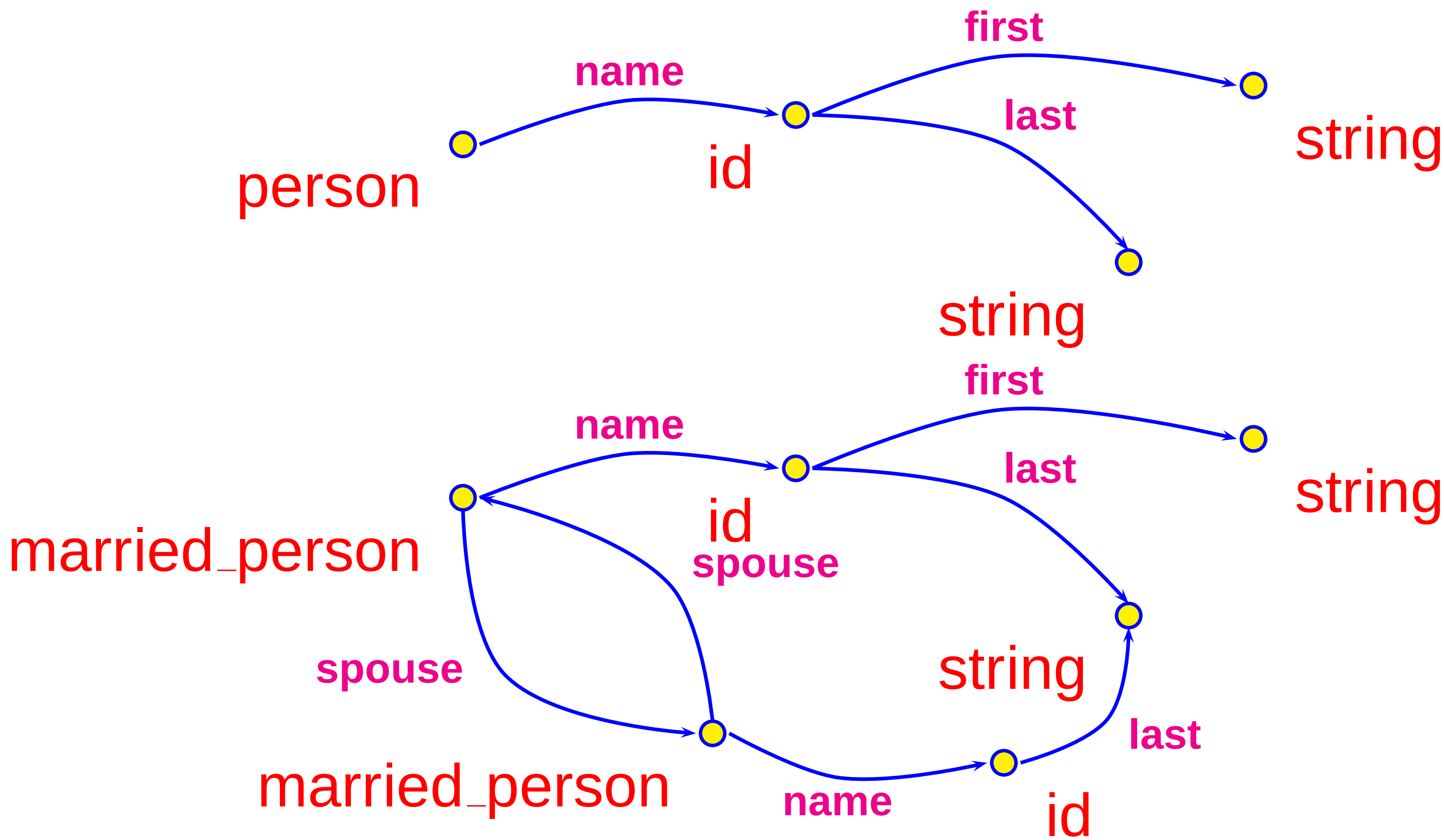
Given an order-consistent $\mathcal{OSF}$ theory Θ, normalize any term of sort s taking into account the $\mathcal{OSF}$ constraints $\Theta(s)$.

Theorem $\mathcal{OSF}$ theory unification is undecidable.

However... there is an algorithm such that:

- inconsistent terms are always normalized to $\perp$ in finitely many steps;
- normalization can perform $\mathcal{OSF}$ constraint inheritance from the theory lazily;
- there is an efficient algorithm which is complete for a large class of $\mathcal{OSF}$ theories;
- only one rule completes it (and may cause divergence).

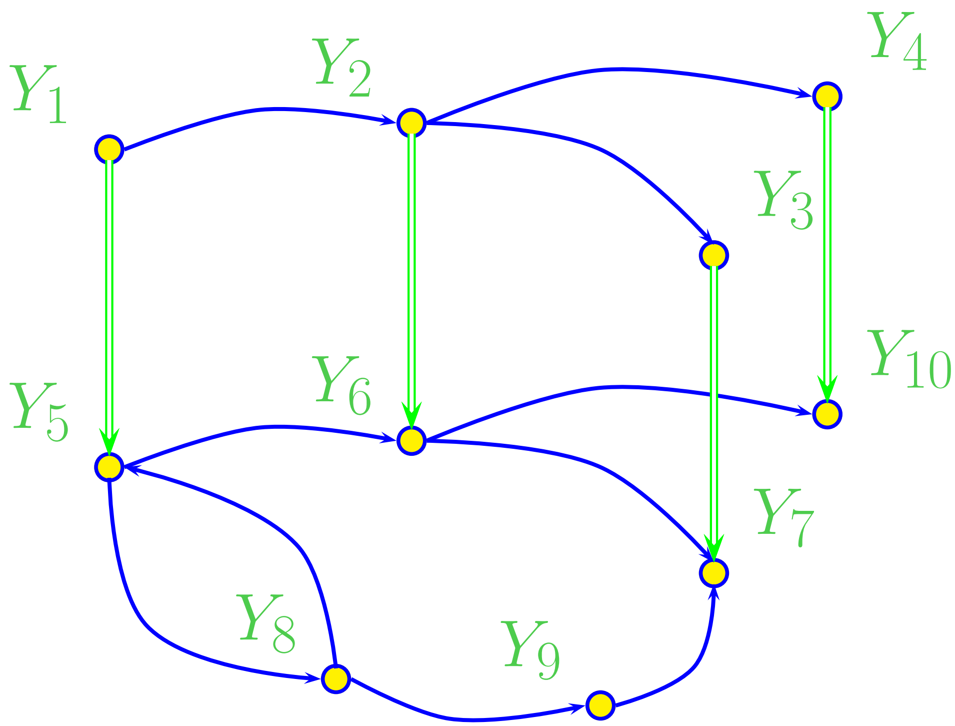
$\mathcal{OSF}$ THEORY UNIFICATION (EXAMPLE)



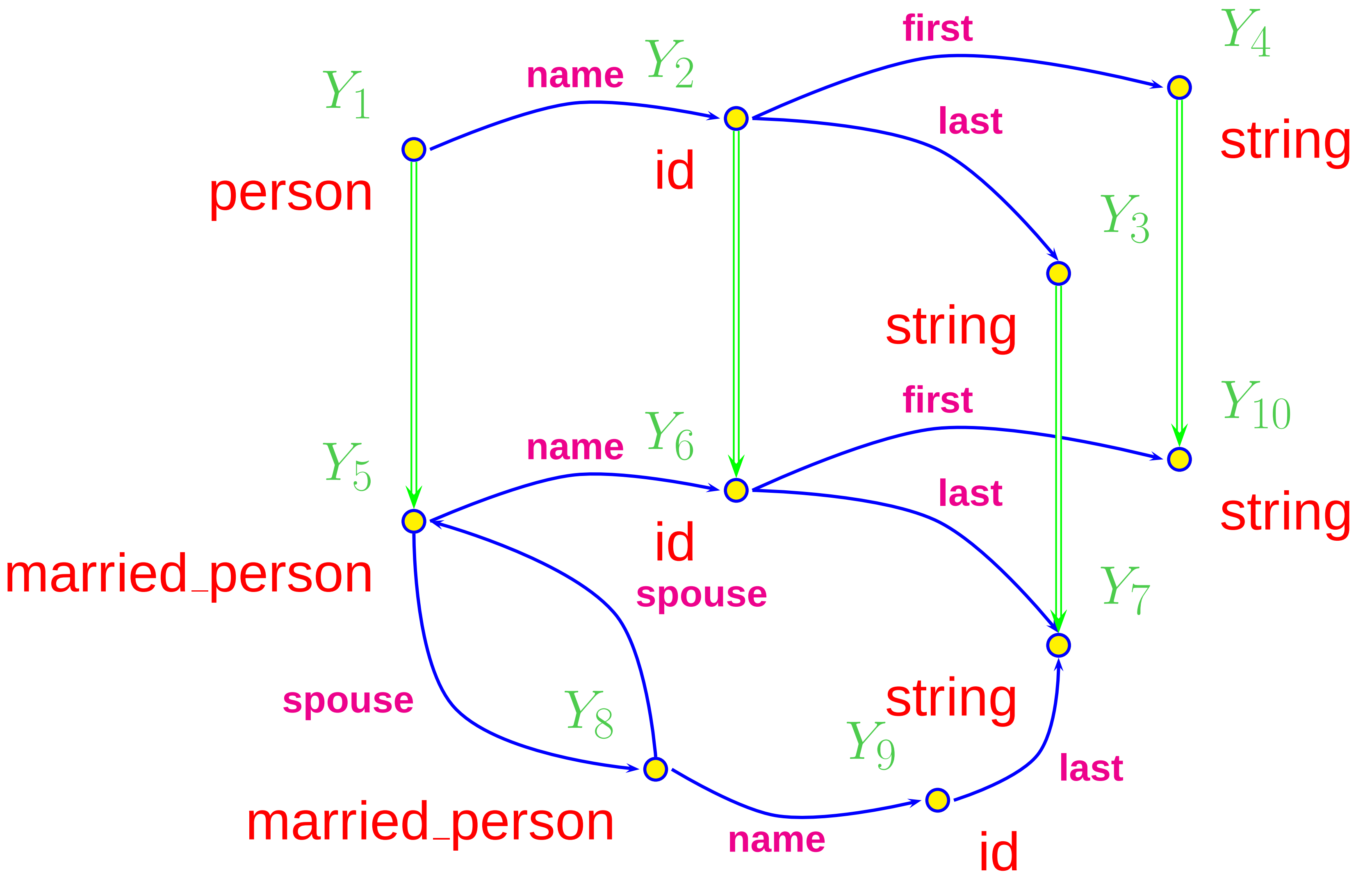
$\mathcal{OSF}$ THEORY UNIFICATION

The fact that an $\mathcal{OSF}$ theory is order-consistent yields an endomorphic mapping of theory variables.

In particular, the sort ordering $\leq$ and the GLB operation $\wedge$ extend homomorphically to all theory variables.



$\mathcal{OSF}$ THEORY UNIFICATION



Normalizing:

$$
\begin{array}{ll}
 & P : person(name \Rightarrow \top(last \Rightarrow \text{“Smith”})) \\
\& & P : married\_person(spouse \Rightarrow Q) \\
\& & Q : person(name \Rightarrow id(last \Rightarrow S))
\end{array}
$$

yields, among other things:

$$
\begin{array}{ll}
 & P : married\_person \\
\& & Q : married\_person \\
\& & S : \text{“Smith”} \ldots
\end{array}
$$

(0) Frame Allocation

$$\frac{\Gamma \qquad \vdash X : s \;\&\; \phi}{\Gamma \bigcup \{\{X\backslash Y_s\}\} \vdash X : s \;\&\; \phi} \quad \text{if } \forall s' \in \mathcal{S},\ \forall F \in \Gamma :\ X\backslash Y_{s'} \notin F$$

$\mathcal{OSF}$ THEORY UNIFICATION (EMPTY THEORY)

(1) Sort Intersection

$$\frac{\Gamma \bigcup \{\{X \backslash Y_{s'}\} \cup F\} \quad \vdash \; X : s \;\&\; X : s' \;\&\; \phi}{\Gamma \bigcup \{\{X \backslash Y_{s \wedge s'}\} \cup F\} \vdash \; X : s \wedge s' \;\&\; \phi}$$

(2) Inconsistent Sort

$$\frac{\Gamma \bigcup \{\{X \backslash Y_{\perp}\} \cup F\} \vdash \; \phi}{\emptyset \qquad \vdash \; \perp}$$

$\mathcal{OSF}$ THEORY UNIFICATION (EMPTY THEORY)

(3) Variable Elimination

$$\frac{\Gamma \vdash X \doteq X' \ \& \ \phi}{\Gamma[X'/X] \vdash X \doteq X' \ \& \ \phi[X'/X]}$$

if $X \neq X'$
and $X \in \textbf{\textit{Var}}(\Gamma) \cup \textbf{\textit{Var}}(\phi)$

(4) Feature Functionality

$$\frac{\Gamma \vdash X.\ell \doteq X' \ \& \ X.\ell \doteq X'' \ \& \ \phi}{\Gamma \vdash X.\ell \doteq X' \ \& \ X' \doteq X'' \ \& \ \phi}$$

(5) Feature Inheritance (if $\ell(Y) = Y'$ and $X'\backslash Y' \notin F$)

$$\frac{\Gamma \bigcup \{\{X\backslash Y\} \cup F\} \quad \vdash \phi \;\&\; X.\ell \doteq X'}{\Gamma \bigcup \{\{X\backslash Y, X'\backslash Y'\} \cup F\} \vdash \phi \;\&\; X.\ell \doteq X' \;\&\; X' : \textbf{\textit{Sort}}(Y')}$$

(6) Frame Merging

$$\frac{\Gamma \bigcup \{\{X\backslash Y_s\} \cup F, \{X\backslash Y_{s'}\} \cup F'\} \vdash \phi}{\Gamma \bigcup \{\{X\backslash Y_{s\wedge s'}\} \cup F \cup F'\} \quad \vdash \phi}$$

$\mathcal{OSF}$ THEORY UNIFICATION (NON-EMPTY THEORY)

(7) Frame Reduction

$$\frac{\Gamma \bigcup \{\{X\backslash Y, X\backslash Y'\} \cup F\} \ \vdash \ \phi}{\Gamma \bigcup \{\{X\backslash Y\} \cup F\} \ \vdash \ \phi} \quad \text{if } Y \leq Y'$$

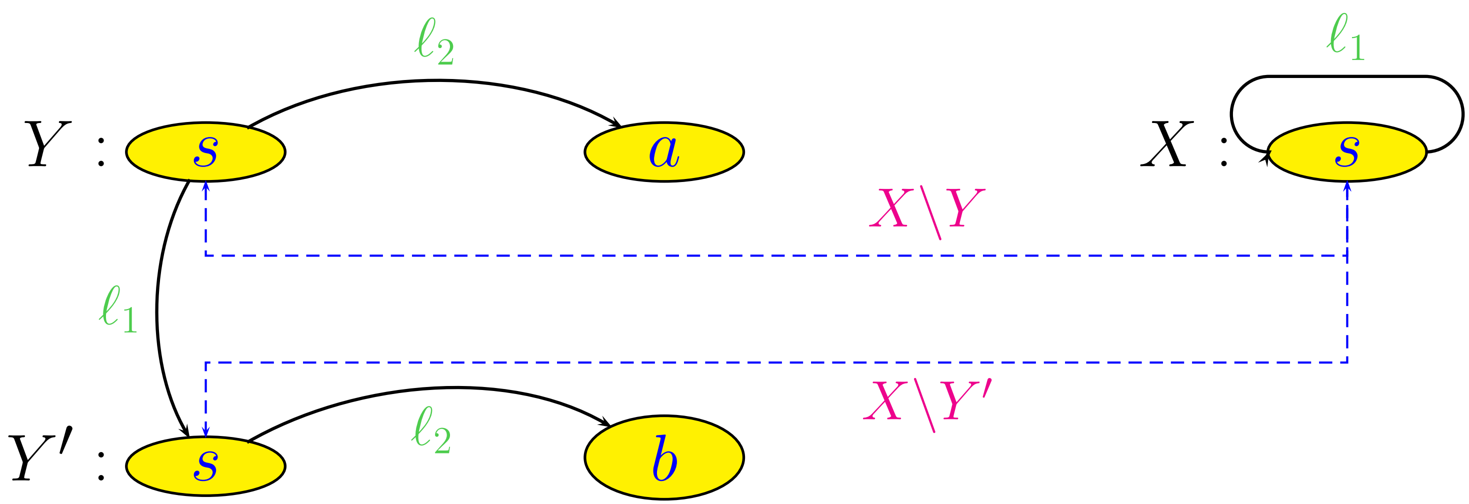
(8) Theory Coreference

$$\frac{\Gamma \bigcup \{\{X\backslash Y, X'\backslash Y\} \cup F\} \ \vdash \ \phi}{\Gamma \bigcup \{\{X\backslash Y\} \cup F\} \ \vdash \ \phi \ \& \ X \doteq X'}$$

(9) Theory Feature Completion

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash X.\ell \doteq Z \;\&\; \phi}$$

if $X \backslash Y \in F$ for some $F \in \Gamma$
and $X \backslash Y' \in F'$ for some $F' \in \Gamma$
and both $\ell(Y)$, $\ell(Y')$ exist
and Z is new



CONCLUSION

We have overviewed a formalism of objects where:

- “real-life” objects are viewed as logical constraints
- objects may be approximated as set-denoting constructs
- object normalization rules provide an efficient operational semantics
- consistency extends unification (and thus matching)
- this enables rule-based computation (whether rewrite or logical rules) over general graph-based objects
- this yield a powerful means for effectively using ontologies

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Thank You For Your Attention !